

Physics 53

Energy 2

Things are more like they are now than they ever were before.
— Dwight D. Eisenhower

Conservative forces

In general the work done by a force depends on the *path* taken from the initial position to the final one, as well as on the location of those points. But, as we have seen, there are some forces for which the work depends only on the endpoints, *not* on the path. These are called **conservative forces**.

Consider a trajectory which ends where it began, forming a closed curve. The work done by a conservative force for such a path must be zero. We therefore have two equivalent ways to define a conservative force:

Conservative force: two definitions	$\int_1^2 \mathbf{F} \cdot d\mathbf{r}$ is independent of path
	$\oint \mathbf{F} \cdot d\mathbf{r} = 0$

In the second case, the small circle around the integral sign denotes a line integral following a *closed curve*, one that ends where it began.

Potential energy

Suppose a conservative (net) force moves a particle from point 1 to point 2, during which time it does positive work, so the kinetic energy increases. Now suppose the same force then moves the particle back to point 1. The work done in this part must be exactly the negative of the other, because the *total* work done by a conservative force in a round trip is zero. The particle's kinetic energy thus decreases back to its original value. Is kinetic energy “created” by the force in the trip from 1 to 2 and then “destroyed” in the return trip? It seems more useful, since this process could be repeated indefinitely, to imagine that the “lost” kinetic energy during the trip from 2 to 1 is *stored* somewhere and can be returned to the particle during its next trip from 1 to 2.

The idea of storing energy is the basis of **potential energy**. It is one of the most important concepts in physics. It is called “potential” because it has the possibility of being converted back into kinetic energy.

Where is the energy stored? That turns out to be a very deep question, to which we will return when we discuss the gravitational field.

Potential energy, denoted by U , is associated with the conservative force, i.e., with the interaction between the particle and its environment. *It is a property of the system of interacting objects.* It is not (like kinetic energy) a property of any single particle.

To make the idea quantitative, we introduce a definition:

Potential energy	$U(2) - U(1) = -\int_1^2 \mathbf{F} \cdot d\mathbf{r}$
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Here, as usual, 1 and 2 are the positions of two points in space.

This definition of potential energy specifies only its *change* from one point to another, but not its value at either point. This leaves things a bit ambiguous, since if U satisfies the definition then $U + (\text{any constant})$ also satisfies it. To specify U at any point precisely, we must choose the arbitrary constant. There are usually convenient choices.

Ordinarily one makes a choice of the point (or points) in space where $U = 0$. This choice has no real physical significance, and is a bit like choosing where to put zero on a scale of temperature.

The definition says the change in U is the *negative* of the work done by the force. If the force does positive work (increasing the kinetic energy) it decreases the potential energy by the same amount. The change in one form of energy is accompanied by an opposite change in the other. This is an important property of conservative forces.

Conservation of mechanical energy

We have been talking only about conservative forces. But in the macroscopic world there are forces which are *not* conservative. Kinetic friction is a clear example. It always does negative work — the force is always opposite to the motion — so the work done depends on the path taken, and for any path, closed or not, it is negative, never zero.

Let us write the total force as a sum of the conservative and non-conservative forces:

$$\mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{non-cons}} + \mathbf{F}_{\text{cons}}.$$

(Each term on the right represents the *total* of the forces of that type.) Now consider the *total* work done by all forces as the particle moves from point 1 to point 2:

$$W_{\text{tot}}(1 \rightarrow 2) = W_{\text{non-cons}}(1 \rightarrow 2) + W_{\text{cons}}(1 \rightarrow 2) = W_{\text{non-cons}}(1 \rightarrow 2) - [U(2) - U(1)].$$

Here U represents the *total* potential energy of all conservative force. We have used the fact that the work done by conservative forces is minus the change in potential energy. Now by the work-energy theorem, the total work is always the change in K , so we have

$$K(2) - K(1) = W_{\text{non-cons}}(1 \rightarrow 2) - [U(2) - U(1)].$$

This can be rearranged:

$$[K(2) + U(2)] - [K(1) + U(1)] = W_{\text{non-cons}}(1 \rightarrow 2).$$

This shows that the *sum* $K + U$ is changed *only* by the work done by non-conservative forces. We introduce a new quantity:

Total mechanical energy	$E = K + U$
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We have arrived at an important result:

Change in mechanical energy	The change in total mechanical energy E of a system is equal to the total work done by non-conservative forces.
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If there is *no* work done by non-conservative forces, there is *no change* in E , and we have another important and very useful result:

Conservation of mechanical energy	If only conservative forces do work the total mechanical energy E is conserved.
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This is one of the fundamental principles of classical physics.

The trick introduced by defining power, work and kinetic energy turns out to be very useful because the work done by non-conservative forces is often zero, negligible, or can be calculated explicitly.

Which forces are conservative?

There is a general mathematical criterion for determining whether a force is conservative. It can be applied to show that the following are conservative forces:

Force	Potential energy	Notes
Constant force	$U(\mathbf{r}) = -\mathbf{F} \cdot \mathbf{r}$	The force must be constant in magnitude and direction.
Gravity near earth's surface	$U(y) = mgy$	Here y is positive upward and $U(0) = 0$.
1-D force: $\mathbf{F} = F(x) \cdot \mathbf{i}$	$U(x) - U(x_0) = -\int_{x_0}^x F(x') dx'$	
Elastic force: $F(x) = -kx$	$U(x) = \frac{1}{2} kx^2$	A special case of the 1-D force.
Central force: $\mathbf{F}(\mathbf{r}) = f(r) \cdot \mathbf{r}$	$U(r) - U(r_0) = -\int_{r_0}^r f(r) \mathbf{r} \cdot d\mathbf{r}$	Universal gravitation is an example.

Obtaining the force from the potential energy

Can we find the conservative force if we are given U ? The answer is yes.

Take the one-dimensional case where $\mathbf{F} = (F(x), 0, 0)$ as an example. In that case the fundamental theorem of calculus can be used to show that

$$F(x) = -\frac{dU}{dx}.$$

The conservative force is the negative of the slope of its potential energy.

In 3-D each component of the force (for example F_x) is equal to the negative of the slope of U in that direction (for example the x direction). The corresponding derivatives are called *partial* derivatives.

Since F determines U (up to a constant), and U determines F , it follows that these two ways of describing a conservative interaction are equivalent. Both are useful and each is preferred in certain cases. The fact that U is a scalar while \mathbf{F} is a vector makes it easier to use U in many cases.

Potential energy plots

In cases where only conservative forces do work, so that E is conserved, it is useful to make a plot of the (total) potential energy function. We will illustrate this in the one-dimensional case discussed above, where U is a function only of x .

Consider the curve shown of the potential energy for some conservative force.

At x_1 U has a minimum and at x_2 it has a maximum. At both points the slope is zero, so the force on the particle is zero at those points. If the particle is placed at either point at rest, it will *remain* at rest. These are points of **static equilibrium**.

But these points differ from each other in an important respect. A particle at rest moved slightly away from x_1

experiences a force directed back toward that point, so remains in that vicinity.

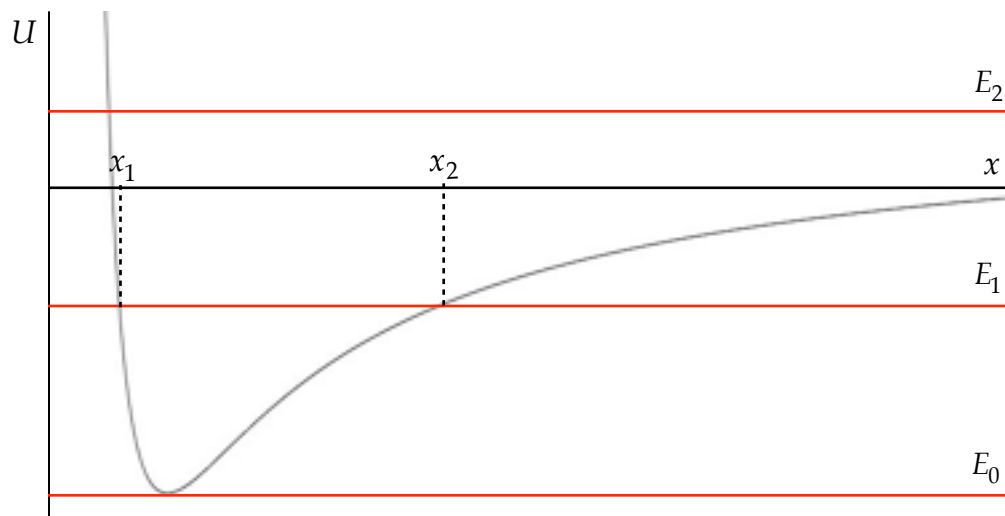
The equilibrium at x_1 is **stable**.

However, a particle moved slightly away from x_2 experiences a force away from that point and will move farther away. This equilibrium is **unstable**.

The rule is simple:

A minimum in U is a point of stable equilibrium, while a maximum in U is a point of unstable equilibrium.

Now we apply conservation of total mechanical energy E . Shown is a typical potential energy for interaction between two atoms in a solid. The red lines represent three different possible (constant) values of the total energy.



Since $K = \frac{1}{2}mv^2$ is positive, and since $K = E - U$, it is not possible to have $E < U$, so any values of x that violates this represent a *forbidden region* for the motion of a particle. A point at which $U = E$ is a point at which the particle is at least momentarily at rest.

If the energy is E_0 there is only a single value of x that is allowed; the particle is at rest at that location. (For this to happen in a real solid the temperature would need to be absolute zero, which is impossible.)

If the energy is E_1 values of x between x_1 and x_2 are allowed; the particle oscillates back and forth, coming to rest and reversing direction at those two points. They are called *turning points* and the motion is a *bound* oscillation between them. (In a real solid this oscillation is called thermal motion.)

For energy E_2 there is only one turning point (at x slightly less than x_1); values of x greater than that are allowed, but because there is no turning point to the right the motion is *unbound*. If the particle comes in from the left, it will stop and reverse motion at the turning point, move off to the right and never return. (This might represent an elastic collision between two atoms.)

In one dimension, location of the points of equilibrium and the turning points is fairly simple, showing why it is useful to make plots of potential energy.

Analysis in terms of energy conservation allows us to discuss the particle's motion as a function of its *position*, but not of *time*. It is nonetheless useful and relatively easy to carry out.